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Discounted optimal growth in a two-sector RSS model: a further geometric investigation*

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Abstract. The geometric apparatus of Khan–Mitra (Adv. Math. Econ. 8:349–381, 2006; Jpn. Econ. Rev. 58:191–225, 2007) enables an identification of a tripartite (*inside-borderline-outside*) distinction for discounted Ramseyian optimality in the 2-sector RSS model and to obtain the following results: (a) parametric ranges of the discount factor for which the check-map is the optimal policy function, (b) necessary and sufficient conditions for the existence of stable optimal 2-period cycles, (c) absence of 3-period cycles in the *borderline* case, and (d) existence of unstable

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3-period cycles in a canonical instance of the *outside* case. The geometry is shown to have more general interest and relevance for future work.

Key words: 3-period convergence, 3-period cycles, Attracting 2-period cycles, McKenzie bifurcation, Optimal policy correspondence, RSS model, Trapezium, Trapping square, Tripartite categorical distinction

1. Introduction

In Khan–Mitra [8], the authors presented a comprehensive analysis of (undiscounted) Ramsey optimality in a special case of a model due to Robinson, Solow and Srinivasan, the so-called 2-sector RSS model, through the identification of a parameter ξ that was interpreted as the marginal rate of transformation of capital from one period to the next with zero consumption.[1] Such a parameter was not identified in earlier work,[2] and by seeing it as the slope of the so-called MV line in a today–tomorrow diagram familiar to students of the general theory of intertemporal resource allocation, the authors relied on a 1970 theorem of Brock's to present a geometric apparatus that revolved around this MV line.[3] Two additional lines were identified: the so-called OD line of slope $(1-d)$, d the rate of depreciation, designating production plans in which the investment-goods sector is shut down; and a VL line, again of slope $(1-d)$, designating production plans in which only

[1] As an anonymous referee and the Editor emphasized, this introduction is not reader-friendly, and assumes a familiarity with the basic geometry of the 2-sector RSS model developed in [5, 8]. Rather than reproduce the analysis of the two papers here, we refer the reader to the six-paragraph recapitulation of the geometry for the undiscounted 2-sector RSS case, originating in [8], in [5, Sect. 3]. Section 4 of the latter paper also delineates how the basic constructions apply, essentially without change, to the discounted case. This being said, in the subsequent section on the basic model, we add some additional explanation to help the reader.

[2] The earlier work that we have in mind concerns the version of the RSS model in continuous time; see Siglitz [14, 15], Cass–Stiglitz [1], and references to the work of Okishio and Joan Robinson. The discrete-time version, as is being anlyzed here, was first presented in [3].

[3] The MV line is the concrete manifestation in the 2-sector RSS model of the concept of a von-Neumann facet, familiar to readers of Lionel McKenzie: the locus of input–output plans that have zero-value loss relative to the golden-rule stock and at the golden-rule prices. As detailed in Footnotes 6 and 14 below, and in the references in Footnote 1 above, this line is also a full-employment no excess-capacity line, and a benchmark-line on which all the indifference curves are pegged. As such it has a triple identity.

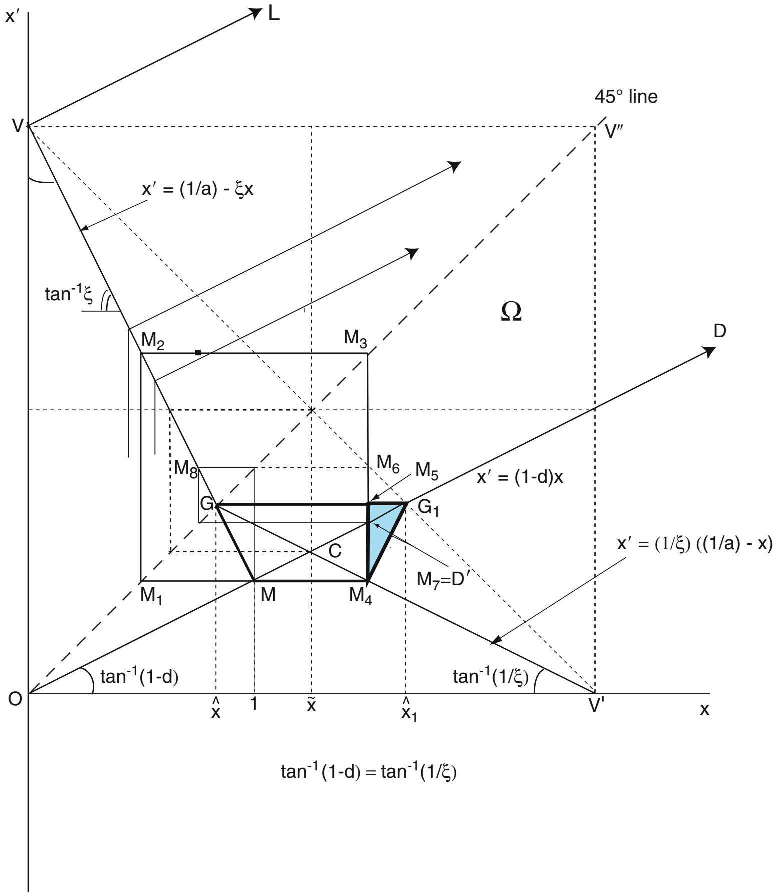

Fig. 1 Basic geometrical benchmarks of the 2-Sector RSS model: the case $\xi(1-\mathrm{d}) = 1$ or $\mathrm{a} = \xi/(1+\xi^2)$ or $(1/\mathrm{a}) = (1-\mathrm{d}) + (1-\mathrm{d})^{-1}$

this investment-goods sector is operative.[4] This apparatus was used in the familiar cobweb setting to identify the optimal policy correspondence. Such a correspondence reduces to a function, the so-called *pan-map,* in the case $\xi > 1$, and guarantees convergence to the golden-rule stock in a finite number of

[4] The reader is referred to Fig. 1 for all the geometrical references in this introduction. For orientation, and in the light of Footnote 2 above, the reader should note that the square $M_1M_2M_3M_4$ in Fig. 1 corresponds to the square M_1QMP in [8, Fig. 13] and to the square $M_1M_2M_3M_4$ in [5, Fig. 5], and to the dotted squares with vertex M_1 in [5, Figs. 4 and 8]. Also see the second and third paragraphs of Sect. 2 below.

periods, though for capital-poor economies, in a precisely-delineated sense, the convergence is not monotonic. In the case $\xi < 1$, the correspondence is again a function, the so-called *check-map,* which again guarantees convergence: monotonic for the subcase $-1 < \xi \leq 0$, but constituted by damped oscillations for the subcase $0 < \xi < 1$. It is only in the remaining case when $\xi = 1$ that Ramsey-optimality yields indeterminacy and a policy correspondence that includes not only the pan- and check-maps but also a triangle they enclose between them, henceforth the *pan-check correspondence.* This result is the only one known to us in the literature on optimal growth theory where a program making higher value-losses than another may nevertheless be optimal. A theory of undiscounted dynamic programming is formulated in Khan–Mitra [6] to provide an analytical demonstration of these results.[5]

In Khan–Mitra [5], this geometric apparatus is extended to the discounted case. It is shown that it is only the case $\xi > 1$ that proves recalcitrant to analysis. For $-1 < \xi < 1$, the transition dynamics are *identical* between the discounted and undiscounted cases, and for the particular value of $\xi = 1$, it is only the indeterminacy exhibited in the optimal policy correspondence that is now eliminated by an operative transversality condition. This is simply to say that the check-map is the optimal policy function for all values of the discount factor, including the value $\rho = 1$. However, for $\xi = (1/(1-d))$, a particular instance within the sub-case $\xi > 1$ that translates into a mutual perpendicularity of the MV and OD lines, it was shown that for all $\rho > (1/\xi)$, the optimal policy function is precisely the pan-map as in the undiscounted case, and that for all $\rho < (1/\xi)$, the optimal policy function is precisely the check-map, with the indeterminacy reappearing in the shape of the pan-check correspondence for $\rho = (1/\xi)$. Furthermore, the check-map isolates a non-negligible continuum of initial capital stocks that generate 4-period cycles. Satisfying as these results are as a vindication of a Fisherian equilibrium, they deal only with the one point within the parametric range; in short, a single instance of the model. In a subsequent, non-geometric and analytic substantiation of the role played by $1/\xi$ through the theory of discounted dynamic programming was presented in Khan–Mitra [9] and referred to as a "folk-theorem" revolving around the McKenzie bifurcation.

However, in this charting of discounted optimal growth in the two-sector RSS model, another set of results pertaining to optimal topological chaos deserves mention. In Khan–Mitra [4], the result that the optimal policy correspondence is a continuous function for all $\rho < 1/(\xi + (1-d)) \equiv a$,[6]

[5] It is perhaps worth pointing out that this analytical demonstration concerns the case $\xi \neq 1$; for the non-generic case of $\xi = 1$, the authors rely on a synthesis of value-loss methods going back to Radner [13], and built on by Brock.

[6] As we shall see below, the parameter ξ is defined by $((1/a) - (1-d))$, where a is the amount of labor required to make a single machine. Hence $a = (1/(\xi + (1-d)))$.

is used to show the existence of optimal topologically-chaotic trajectories in the particular instance $(\xi - (1/\xi))(1 - d) = 1$. This finding appeals to results on turbulence. Furthermore, as a byproduct of a construction presented in Khan–Mitra [7], a similar finding of the existence of optimal (topologically) chaotic trajectories in the particular instance $\xi = (1 + (1/(1 - d)))$ is established as a direct consequence of the Li–Yorke theorem. On relying on analytical results of [4],[7] a simple and unified geometric argument can be presented for both instances. The underlying methodological premise behind this work is worthy of emphasis: it is simply that other than continuity, the shape of the optimal policy correspondence can remain completely unknown. However, this raises the natural question as what the optimal policy function really is at least in these two cases.[8]

We answer this question here through a geometric analysis that proceeds beyond earlier work in several important directions. The first point to be emphasized is that the sustained application of the geometric apparatus to these two cases reveals a categorization totally missed in earlier work. It allows us to perceive the instance of the model studied in [4] as a *borderline* case, separating an *inside* case from an *outside* one. In particular, it isolates a trapezium as a kernel of interest in the 2-sector RSS geometry for the case $\xi > 1$, the upper triangle of this trapezium being precisely the triangle of the pan-check correspondence referred to above. In Fig. 1, this trapezium is given by GMM_4G_1. The vertices of this trapezium are worthy of notice: in addition to the kink M of the check-map, they are the golden-rule stock indicated by G, and the capital stock represented by G_1 from which a shutting down of the investment-goods sector leads to the golden-rule stock in the subsequent period. Figure 2 represents two cases depending on whether the point M_5 falls strictly *inside* or *outside* the interval GG_1. Parameters of the model that give rise to the two possibilities will be respectively referred to as exhibiting the *inside* case, or the *outside* case. The case where G_1 equals M_5 will be referred to as the *borderline* case.[9] As the alert reader will undoubtedly note, Fig. 2 is simply a visual representation of the analytical comparison of the two angles $\tan^{-1}(1 - d)$ and $\tan^{-1}(1/\xi)$, these being none other than the respective slopes of the OD and the dual MV lines.

[7] From here on, having mentioned the relevant earlier work of the authors by name, we adopt the convention of referring to an item by its number in the list of references.

[8] This work has now received extension and elaboration in [10]. The construction, originally presented in [7] is incorporated in ongoing work by Khan and Adriana Piazza.

[9] For a further orientation, and as an exercise, the reader may wish to look ahead at Figs. 7 and 8, along with Fig. 1, and determine for herself where they fall within the tri-partite categorization being proposed in this paragraph.

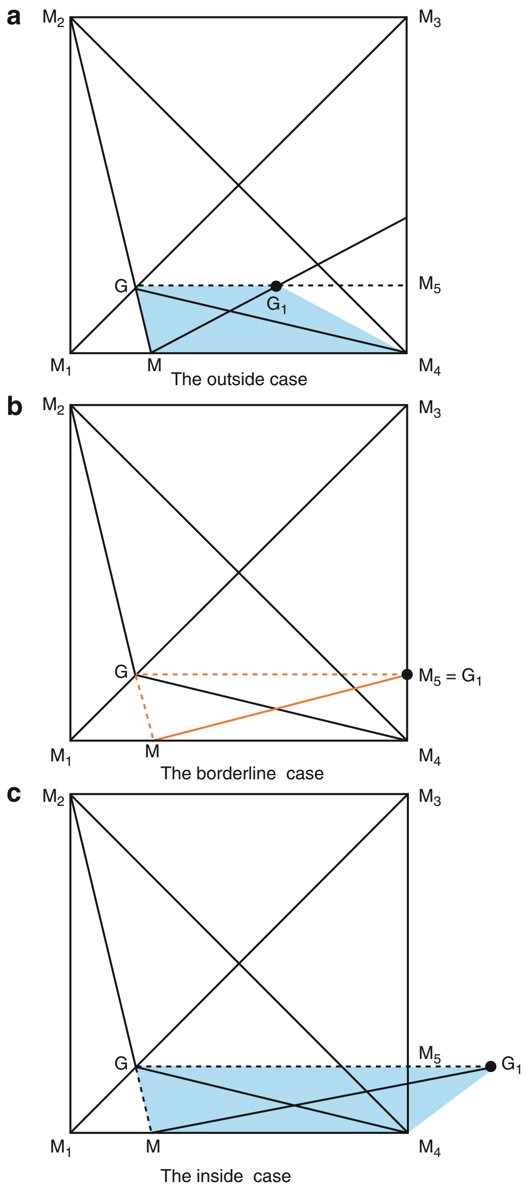

Fig. 2 The tripartite categorization: placement of M_5 in the interval GG_1

This tri-partite categorization can also be presented solely in the vernacular of the dynamics of the check-map. Under the parametric case $(\xi - (1/\xi)) = 1/(1-d)$, referred to above and discussed in [4], the check-map is one under which the unit capital stock converges to the modified golden-rule stock precisely after *two* periods by virtue of the fact that the capital stock in the second period is represented by the point $M_5 = G_1$, which is to say that M_5 lies precise on the endpoint (border) of the interval GG_1. The two instances of the model considered in [5], and discussed above under the parametrizations $\xi = 1$ and $\xi = (1/(1-d))$, can both be seen as *inside* cases. In each instance, M_5 lies *inside* the interval GG_1. Finally, under the parametrization $\xi = (1 + (1/(1-d)))$, M_5 lies *outside* the interval GG_1, and it is thereby revealed to be an outside case.[10]

Furthermore, as we emphasize below, it is the fact that this trapezium is an isosceles trapezium in the case $\xi(1-d) = 1$, the case considered earlier in [5], that is responsible for the particular symmetry of this case. Indeed, the essential contribution of the analysis of this case presented earlier hinged on the identification of a line dual to the MV line: the GV' line in Fig. 1. The two lines are dual in the specific sense that the sum of their slopes constitute a right angle. It was argued that this dual MV line serves as important a role in the theory of the 2-sector RSS model as the MV line itself: whereas the intersection of the MV line and the 45°-line yields the golden-rule stock (and also the modified golden-rule stock in the discounted case), the intersection of the dual MV line and the OD line yields an optimal 2-period cycle for all the discount factors for which the check-map is the optimal policy function.[11] This dual MV line was seen to isolate a square of analytical relevance, and what we now see, and establish in the sequel, is that the both the number of bifurcations and the transition dynamics hinge crucially on how the trapezium GMM_4G_1 relates to the square $M_1M_2M_3M_4$.

With this borderline-inside–outside distinction at hand, we can turn to the question posed above as regards the optimal policy correspondence for the two instances of the 2-sector RSS model. The analysis of the first case mimics that obtained in [5] in that we again obtain the pan- and check maps and the pan-check correspondence for the identified values of the discount factor. The second case, however, yields a surprise. There are now two instances of indeterminacy, two points of bifurcations of the discount factor. For the case $\rho = (1/\xi)$, the optimal policy correspondence consists of *two* pan-maps and a trapezium that they enclose; and for the case $\rho = \rho_c < (1/\xi)$, ρ_c to be delineated below, the optimal policy correspondence consists of a second

[10] In keeping with Footnote 9, the parametrization herein discussed first is presented in Fig. 7, the second in Fig. 1, and the third in Fig. 8. For the parametrization, $\xi = 1$, will have to draw the associated figure herself or go to figure in [5, Fig. 8].

[11] And possibly, in some cases, for higher values of the discount factor. Since this possibility does not arise in the two cases on which we focus in this paper, we leave its analysis for the future.

pan-map, the check-map and a corresponding triangle that they enclose. This instance, and its analysis, is important in that it disposes of the conjecture that there is only one McKenzie bifurcation of the discount factor for the 2-sector RSS model and thereby reveals its unexpected richness. Furthermore, we can go beyond these two instances of the 2-sector RSS model, and proceeding within the inside case, establish the existence of a unique McKenzie bifurcation for the parametrization $1 < \xi < (1/(1-d))$.

Once the check-map is identified as an optimal policy function for specific ranges of the discount factor, we can turn to the resulting optimal dynamics. This constitutes the third direction in which the analysis of this paper goes beyond results reported in earlier work. We furnish for all discount factors in the range $\rho < (1/\xi)$, necessary and sufficient conditions for the existence of an optimal *attracting* 2-period cycle in the 2-sector RSS model. We can show that there exists a capital stock $\tilde{x}$ greater than unity to which all optimal programs converge. In the light of the authors' earlier work on optimal chaos referred to earlier, this result therefore establishes the impossibility of such chaotic dynamics for the parametric range under consideration.[12] As far as the two (primary) parametric instances considered in this paper, even though we are still several steps away from a complete analysis of the dynamics of the check-map, we can identify another important ingredient of our geometric apparatus: a line OD_2 with slope $(1-d)^2$ "below" the OD line with slope $(1-d)$.[13] Such a line can be used to diagnose the presence of a 3-period cycle, and it enables us to offer two results: (a) the absence of a 3-period cycle in the borderline case, (b) the instability of the 3-period cycle in the outside case.

The remainder of the paper is as follows. After a specification of the model and the geometric antecedents in Sect. 2, we present the substantive analysis in Sects. 3–5, one section for each parametrization, and with each sub-sectioned into a discussion of the benchmarks, the dynamics and the bifurcations. The third identifies non-degenerate ranges of the discount factor under which the check- and pan-maps are the optimal policy functions, as well as the resulting transition dynamics in these cases. Section 6 ends the paper with some observations oriented to for future work needed for a complete characterization of the optimal policy correspondence in the general setting, and a complete delineation of the optimal dynamics corresponding to it.

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[12] For the importance of 2-period cycles in the theory of optimal growth, see Mitra [12]. Also see [2] for the pervasiveness of cyclical behavior in the Leontief-Shinkai model.

[13] It is the squared term that leads us to name this the OD_2-line; to refer to the OD line as the OD_1-line would surely be excessive pedantry.

2. The Model and Its Geometrical Antecedents

In this section we present the two-sector model, and recall the basic features of the geometrical apparatus presented in Khan–Mitra [5, 8].

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers serving as the time periods. Thus, the *transition possibility set,* Ω, formalizing the collection of feasible production plans (x, x'), the amount x' of machines in the next period (tomorrow) from the amount x available in the current period (today), is given by

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1 - d)x \geq 0 \text{ and } a(x' - (1 - d)x) \leq 1\},$$

where $\mathbb{R}_+$ is the set of non-negative real numbers, $z \equiv (x' - (1 - d)x)$ the number of machines produced in period t, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on reversibility of investment and on the use of labor. The set Ω, as constituted by these two constraints, is portrayed as the "open" rectangle $LVOD$ in Fig. 1.

The preferences of the planner are represented by a linear felicity function, normalized so that its value is identical to the amount of the consumption good. If, for any $(x, x') \in \Omega$, y represents the amount of machines available for the production of the consumption good, given the normalizations adhered to, it also represents the amount of the consumption good that is available. Given the pair $(x, x') \in \Omega$, the stock of machines devoted to the consumption goods sector is given by the correspondence

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1 - d)x)\}.$$

Hence the *reduced form utility function,* $u : \Omega \longrightarrow \mathbb{R}_+$, is given by

$$u(x, x') = \min\{x, 1-a(x' - (1 - d)x)\}.$$

In Fig. 1, the indifference curves of the reduced form utility function $u(\cdot, \cdot)$ are the kinked lines OVL, $1MD$ and the two others shown in between. The first is the minimum, zero-felicity curve, and the second, the maximum, unit-felicity curve, the levels of felicity increasing as the curves move southeast. What is important and well-understood is that the linearity of the felicity function does not imply the linearity of the reduced-form felicity function $u(\cdot, \cdot)$. The reduced-form model is now completely determined by the three parameters (a, d, ρ). The locus of all the kinks of the indifference curves is furnished by the important, and aforementioned, MV line. We remind the reader that these kinks also represent full employment of labor and capital (existing stock of machines), and that therefore we obtain

$$x = 1 - a(x' - (1-d)x) \Longleftrightarrow x' = (1/a) - [(1/a) - (1-d)]x$$
$$\Longleftrightarrow x' = (1/a) - \xi x,$$

where $\xi > -1$ is the slope of the MV line representing the marginal rate of transformation of today's stock of machines into tomorrow's stock, given zero consumption levels, and *the* "sufficient statistic" for the 2-sector RSS model.[14]

An *economy* E consists of a triple (Ω, u, ρ), $0 < \rho \leq 1$ the discount factor, and the following concepts apply to it. A *program from* x_o is a sequence $\{x(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$. A *program* $\{x(t)\}$ is simply a program from $x(0)$. A program $\{x(t)\}$ is called *stationary* if for all $t \in \mathbb{N}$, $(x(t)) = (x(t+1))$. For all $0 < \rho < 1$, a program $\{x^*(t), y^*(t)\}$ from x_o is said to be *optimal* if

$$\sum_{t=0}^{\infty} \rho^t [u(x(t), x(t+1)) - u(x^*(t), x^*(t+1))] \leq 0$$

for every program $\{x(t)\}$ from x_o. A *stationary optimal program* is a program that is stationary and optimal.

We now recall the basic observation in [5] that the modified golden-rule stock is given by the point G, and that it yields the highest utility among all plans in Ω which lie "above" a line with slope $(1/\rho)$, and passing through G. As is by now well-understood, the modified golden-rule stock $\hat{x}$ solves the following problem:

$$u(\hat{x}, \hat{x}) \geq u(x, x') \text{ for all } (x, x') \in \Omega \text{ such that } x \leq (1-\rho)\hat{x} + \rho x'.$$

Since $u(\hat{x}, \hat{x}) > u(0, 0)$, $\hat{x}$ satisfies precisely the definition of the discounted golden-rule stock as in McKenzie [11] and his references.[15] When ρ is unity, this line collapses into the 45°-line, and the analysis in [5] reduces to that in the undiscounted case studied in [8]. More formally, it is the unique plan that satisfies $u(\hat{x}, \hat{x}) \geq u(x, x')$ for all $(x, x') \in \Omega$ such that $x \leq x'$. The distinguishing characteristic of the 2-sector RSS model, already established in [5], is that the golden-rule stock is invariant to changes in the discount factor.

[14] See Footnotes 3 and 6 above, and the economic interpretation of ξ in [5, 8].
[15] See the line RG in [5, Fig. 1].

In summary, the geometric representation of the economy is given as in Fig. 1, by the lines VL and OD, where OV is given by the output–input coefficient $(1/a) > 0$ in the investment-goods sector, and the slopes of the two lines being given by the depreciation rate $(1-d)$, $d \in (0, 1)$. One can now mark out the point M with coordinates $(1, (1-d))$, the point V with coordinates $(0, 1/a)$, and finally the intersection of the MV line with the 45°-line to yield the modified golden-rule stock represented by the point $G = (\hat{x}, \hat{x}) = (1/(1+ad), 1/(1+ad))$.

All this is routine transposition of the geometry for the undiscounted case developed in [8] to the discounted case, and draws on the important fact that the modified golden-rule stock G is independent of the discount factor ρ. What is new to [5] is the emphasis on two squares ($OVV''V'$ and $M_1M_2M_3M_4$ in Figs. 1, 7 and 8), and on the line GV' interpreted as dual to the line MV. It is the intersection of GV' with OD that yields the point C, the capital stock $\tilde{x}$ generating a 2-period cycle. In this context, it ought to be borne in mind that this intersection is the unique 2-period cycle, an observation stemming from the fact that a 2-period cycle is necessarily (and sufficiently) given by a square with a vertex on the OD line, and with a side given by the length of this point to the 45°-line; see Fig. 3.[16]

A complete analysis of the cases $\xi = 1$ and $\xi(1-d) = 1$ was presented in [5]. The essential geometric observation in [5] relates to the trapping square $M_1M_2M_3M_4$: for any point on the left diagonal below the center of the square, such as G in Figs. 2 and 3, consider the lines GM_4 and GM_2. They are dual in the specific sense that their slopes are commensurable i.e. the slope of one is reciprocal to that of the other. This leads to the property that any point in the diagonal segment GM_1 has the property that it is a vertex of a square with its sides parallel to M_1M_2 and M_1M_4. In Fig. 3, the points m_2, m_3, m_4, m_5, m_6, and indeed the point M_1 itself, can all be seen as such vertices with their corresponding vertices on GM_4 being n_2, n_3, n_4, n_5 and M_4 respectively.

This has a substantive consequence, already noted in [5] but being fully exploited in this paper, that any plan chosen on the segment MG_1 of the line OD, say m_1 in Fig. 3, by virtue of the square with the left lower vertex m_4, results in a capital stock determined by the plan n_4 on GM_4, (labeled as n_{42} on the line MD'); and furthermore, by completing the square with right lower vertex n_2 and side (m_2, n_2), results in the plan m. Thus the arbitrary plan m_1 gives rise to the three plans m, n_4 and n_2. As further illustration, the plan M results in a capital stock determined by D', and the plan m results in the capital stocks determined by the pair (m_2, m_1) the latter determined via n_2. The particular specification $\xi(1-d) = 1$ (in Fig. 1) results in the

[16] We owe this observation to David Wiczer.

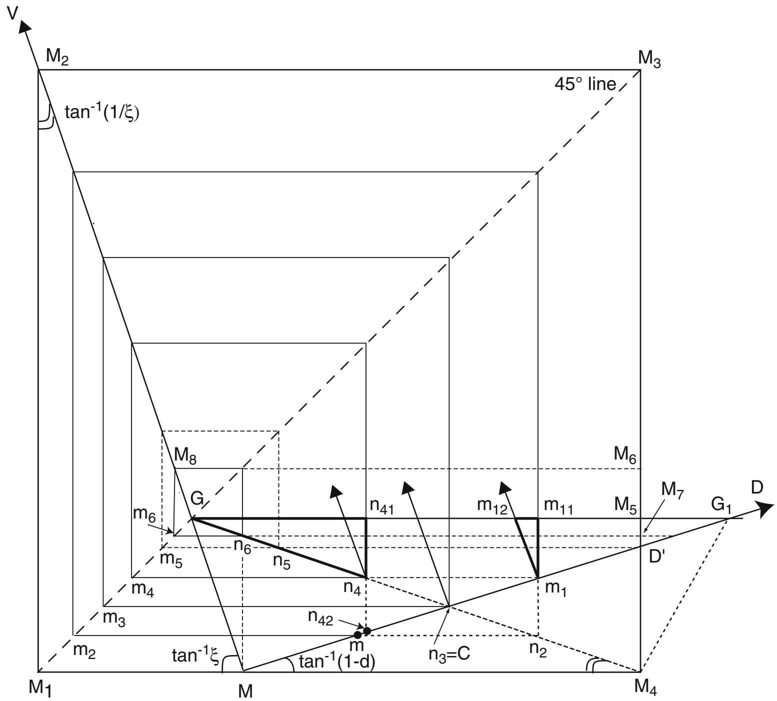

Fig. 3 The "trapping square" $M_1M_2M_3M_4$ of side dξ and the trapezium MM_1GG_1 in the inside case

slope of the OD line being identical to that of the GV' line, resulting in the plans M_7 and D' being identical, and more relevantly to the point being currently emphasized, the sighting of the paired plans (M_1, M_4) as being dual plans; see [5, Sect. 6.5]. If Fig. 3 is redrawn with the particular specification of Fig. 1, the line OD is rotated upward so as to make D' identical to M_7, enabling the plans m and n_4 being on the same vertical, and the quadruple (m, n_2, m_1, n_4) constituting a rectangle and being reduced to the pair (m, m_1) being regarded as dual plans. This duality, in turn, lead to the establishment of a continuum of 4-period cycles in [5].

In the argument recapitulated above, there is of course no presumption that the check-map is an optimal map. That this is indeed so for values of the discount factor $\rho < (1/\xi)$ was one of principal contributions of [5] for the two parametric instances considered therein. The essential idea of the proof is to relate the ratio of value-losses at two different plans to being identical to the ratio of their projections by the corresponding value-loss lines onto a particular horizontal. Thus, in Fig. 3, the ratio of value-losses at (say) the plans at m_1 and m_{11} is given by the ratio Gm_{11}/Gm_{12}.[17] This allows one to completely determine the optimal policy correspondence for all values of the discount factor ρ in the two cases $\xi = 1$ and $\xi(1-d) = 1$. As mentioned above, in either case, the policy correspondence is the pan-map for all ρ greater than $1/\xi$, including unity, the check-map for all ρ less than $1/\xi$, and a correspondence that includes these maps along with the triangle $\triangle GMG_1$; see [5, Figs. 5 and 8]. However, as we shall see in the sequel, this argument relies crucially on their being a continuum of cycles, 2- or 4-period ones in the two respective cases. For the cases under consideration here, we cannot rely on this consideration and a complete characterization thereby remains elusive.

We now turn from this background analysis to move it forward to this paper.

3. The Case $1 < \xi < (1/(1-d))$

This range of parametrization lies between the two polar ones considered in [5], both of which fall within the category of the *inside* case, and it is thus natural to ask for an analysis that we provide in this section.

3.1. The Benchmarks

The benchmark that characterizes the case under consideration is simply the fact that the slope of the MD line in Fig. 3 (also MD' line in Figs. 4–6) is smaller than that of the GM_4 line in absolute value. This can be alternatively expressed as the fact that the point D' lies below M_5 (and M_7), or to get yet another perspective, that the capital stock determined by the plan n_4 in Fig. 3 lies to the right of that determined by the plan m, both corresponding to the arbitrarily-chosen plan m_1.

In terms of previous work, in the case $\xi = 1$, the lines GM_2 and GM_4 in Fig. 3 are collinear, and in the case $\xi(1-d) = 1$, perpendicular. Staying with the latter case, the lines MD' and GM_4 in Fig. 1 have equal slopes (in absolute value), and that the analogue of plan n_4 in Fig. 3 lies on the vertical through m.

3.2. Check-Map Dynamics

It is precisely the exploitation of this benchmark that leads to a complete characterization of the case under consideration. Towards this end, consider

[17] For a detailed discussion of this ratio of value-losses, see Sects. 4 and 6 in [5, Sects. 4, 6].

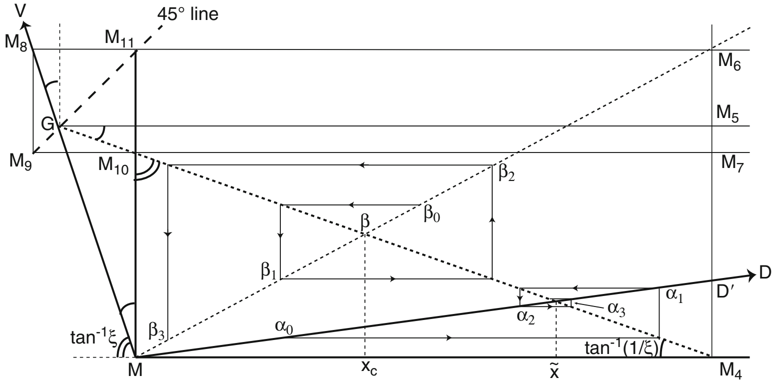

Fig. 4 Stability or instability of a 2-period cycle

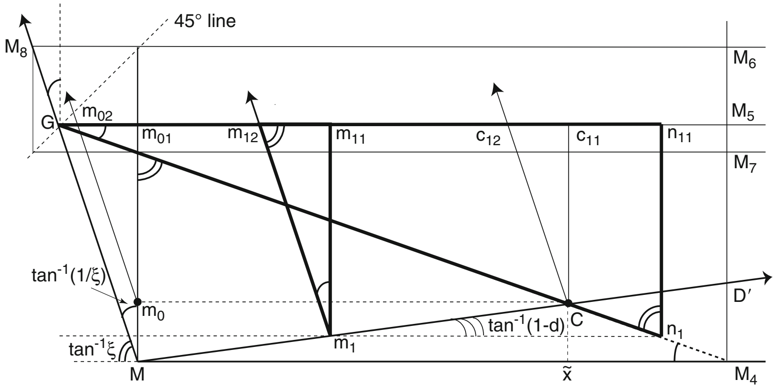

Fig. 5 Determination of ρ_1 in the case $\xi(1-\mathrm{d}) < 1$

Fig. 4 which is focussed on the figure $MM_8M_6M_4$, a subset of the square $M_1M_2M_3M_4$, a third step in progressive magnification. It is now easy to see that any plan on the segment MD', say α_0, by completion of the square, leads to the plan α_1 and thence to the plans $\alpha_2, \alpha_3, \cdots$, eventually converging to the capital stock $\tilde{x}$, and hence to the unique 2-period cycle. Just as the slope of the MV line relative to the 45°-line led to the initial opening into the transitional dynamics of the 2-sector RSS model,[18] it is the slope of the dual MV line relative to the OD line that leads to the opening into the transitional dynamics of the 2-sector RSS model in the case ξ greater than unity. Indeed,

[18] This is *the* observation that established the viability of the geometric engine in [8].

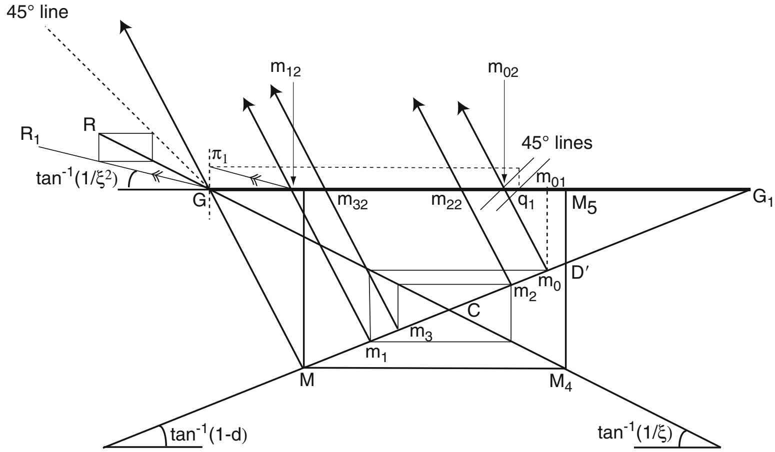

Fig. 6 Determination of ρ_1 in the case $\xi(1 - \mathrm{d}) < 1$

we also indicate in Fig. 4 a situation in which the OD line is given by MM_6, with the corresponding non-attracting 2-period cycle given by x_c. For a proof that any plan, near and unequal to x_c, moves away from x_c, simply follow the plans $\beta_0, \beta_1, \beta_2, \beta_3, \cdots$. In summary, the fact that the OD line has a smaller slope than the dual of the MV line, is a necessary and sufficient condition for the occurrence of an attracting 2-period cycle.[19]

3.3. The McKenzie Bifurcation

We have yet to establish values of the discount factor ρ for which the check map constitutes the optimal policy function. Before turning to this, it is important to be clear why the geometric methods for delineating the optimal policy correspondence developed in [5, 8] do not automatically extend to this case. The point is simply that in the two cases considered in [5], the argument relies crucially on the fact that any plan on the arms M_2M and MG_1 lead to a 2-period cycle (in the case $\xi = 1$), or to a 4-period cycle (in the case $\xi(1 - d) = 1$); see Fig. 1. As such we can compute the value-loss of a path starting from it, and compare its value with the straight-down-the-turnpike path given by the initial plan on the arm GG_1 of the pan-map. Given pervasive convexity (indeed, linearity of the model), the discount factor which brings about the equality of this comparison is enough to pin down the optimal policy function. It is the absence of this feature that requires a substantial extension of the argument.

[19] See Footnote 16 and the text it footnotes.

In the case under consideration, such a comparison can be made for a program starting at C in Fig. 5. We shall refer to programs that converge to the golden-rule stock G in one period as *straight-down-the-turnpike* programs, and the point is to compare a straight-down-the-turnpike path with one that cycles every two periods. To recall the argument made in [5], we need to work with

$$\delta^{\rho}(\tilde{x}, \hat{x}) = \delta^{\rho}(\tilde{x}, (1-d)\tilde{x})(1+\rho^2+\rho^4+\cdots) = \delta^{\rho}(\tilde{x}, (1-d)\tilde{x})(1/(1-\rho^2)), \tag{1}$$

where $\delta^{\rho}(x, x')$ is a single-period value loss of a plan with coordinates (x, x'), and where $\tilde{x}$ is the initial stock at the plan C. And now by the result on the ratio of value-losses mentioned earlier,[20] we obtain that

$$\rho^2 = \frac{\delta^{\rho}(\tilde{x}, \hat{x}) - \delta^{\rho}(\tilde{x}, (1-d)\tilde{x})}{\delta^{\rho}(\tilde{x}, \hat{x})} = \frac{c_{11}c_{12}}{Gc_{11}} = \frac{c_{11}c_{12}}{c_{11}C}\frac{c_{11}C}{Gc_{11}} = \frac{1}{\xi^2}. \tag{2}$$

Now, just as in [5], the reader can check, that for $\rho > (1/\xi)$, the plan c_{11} is optimal, while for $\rho < (1/\xi)$, the plan C is optimal. This completes the the first step of the argument.

However, such a clear-cut comparison is no longer possible programs with initial stocks other than C. The reason is the obvious one that convergence to the plan C from such stocks takes an infinite number of time-periods and hence an evaluation of an infinite series of discounted value-losses. Figure 6 illustrates the issue. We need to show that the discount factor $(1/\xi)$ equates the value-loss of the straight-down-the-turnpike path to the accumulated value-losses of a path that converges to the plan C, which is to say, the path constituted by the plans $m_0, m_1, m_2, \cdots$. From a geometrical point of view, the analogue of Eq. (1) and of Eq. (2) would be one that equates the segment Gm_{01} to the weighted sum of the segments $Gm_{02}, Gm_{12}, \cdots$, the weights respectively being $1, 1/\xi^2, 1/\xi^4, \cdots$. The required procedure then consists of the following steps illustrated in Fig. 6. Obtain the line GR_1 of slope $\tan^{-1}(1/\xi^2)$ from the line GR of slope $\tan^{-1}(1/\xi)$ by completing the rectangle at G through the use of the 45°-line. Next, draw a line at m_{12} parallel to GR_1 with its intersection with the vertical at indicated by the point π_1. Third, obtain, again through the 45°-line at m_{02} the point q_1 where $m_{02}q_1$ equals $G\pi_1$. The second term Gm_{12}/ξ^2 in the weighted sum of value-losses is precisely $m_{02}q_1$. But now the succeeding iterative steps of the argument are clear. We obtain the line GR_2 of slope $\tan^{-1}(1/\xi^4)$ from the line GR_1 of slope $\tan^{-1}(1/\xi^2)$ precisely by a completion of the relevant rectangle as before, and by shifting it to the plan m_{22}, to obtain the point π_2, and finally

[20] See the last but one paragraph of Sect. 2 above, and Footnote 17 for a precise reference to [5].

the point q_2 through a 45°-line at q_1. The reader can see that the segments $G\pi_i$ converge to zero. What is being claimed, and has to be shown through analysis, is that this point is precisely the point m_{01}. This overview of the underlying argument involves a second order non-autonomous linear difference equation.[21] The point is that even though Fig. 6 illustrates the structure of the argument, it cannot clinch it, and thereby brings out the apparent inadequacy of the geometric approach. Whenever the summation of an infinite number of sums is required, geometry is naturally required to defer to analysis.

However, this difficulty can be *bypassed!* We make the plan C, instead of the plan M, the lynch-pin of the argument. What is primarily at issue is that the OD line intersects the square at a point D' below M_5 (see Figs. 3–6), and that it is a specific instance of an *inside* case illustrated in Fig. 2c. It is this that allows the feasibility of a program that begins at M, and converges to the golden-rule stock in the third period, an observation that can be exploited to give a complete geometric characterization of the optimal policy red correspondence in the case under consideration. In terms of an overview, we proceed in three steps: in the context of Fig. 3, and at the discount factor $(1/\xi)$, we show that (a) specified feasible programs starting from the plans m_1 and m_{11} have identical values losses, (b) specified feasible programs starting from plans on the arm MC have identical value losses, (c) use the optimality of the plan C to establish that this common value-losses are indeed the optimal value-losses.

Towards this end, consider the plan represented by the point m_1 in Fig. 3. Certainly the program that begins at the point m_{11} and stays at the golden-rule stock thereafter is feasible, as is the program that begins with the plan m_1, continues on through m_4 to n_{41}, and stays at the golden-rule stock thereafter. We can now determine the value of the discount factor ρ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\delta^\rho(x_m, \hat{x}) = \delta^\rho(x_m, (1-d)x_m) + \rho^2\delta^\rho(x_n, \hat{x})$$
$$\Longrightarrow \rho^2 = \frac{\delta^\rho(x_m, \hat{x}) - \delta^\rho(x_m, (1-d)x_m)}{\delta^\rho(x_n, \hat{x})},$$

where x_m and x_n are the respective initial stocks at the plans m_1 and n_{41}. And now by the result on the ratio of value-losses mentioned earlier,[22] we obtain

$$\frac{\delta^\rho(x_m, \hat{x}) - \delta^\rho(x_m, (1-d)x_m)}{\delta^\rho(x_n, \hat{x})} = \frac{Gm_{11} - Gm_{12}}{Gn_{41}} = \frac{m_{11}m_{12}}{Gn_{41}}.$$

[21] A full analysis of this equation will be presented elsewhere.

[22] See Footnotes 17 and 20, and the text they footnote.

Next, by focussing on $\triangle G n_{41} n_4$ and $\triangle m_{12} m_{11} m_1$, the triangles in bold in Fig. 3, we obtain

$$\frac{m_{12}m_{11}}{m_1 m_{11}} = \frac{1}{\xi}, \quad \frac{Gn_{41}}{n_4 n_{41}} = \xi.$$

Since the segment $n_4 n_{41}$ equals $m_1 m_{11}$, we can eliminate it to obtain that $\rho = 1/\xi$. Again, by appealing to the pervasive convexity of the model, any plan in the segment $m_1 m_{11}$ would be part of an optimal program at the discount factor $(1/\xi)$ if either of the initial two plans were optimal. Furthermore, this argument carries over *verbatim* to any initial plan in the segment $n_3 G_1$.

Next, we turn to initial plans in the segment Mn_3, say the plan m in Fig. 3, or the plan m_1 in Fig. 5. Again, on equating the value-losses from the straight-down-the-turnpike program to those obtained by shutting down the investment sector in the first period, and then going straight-down-the-turnpike once enough capital has been accumulated, we can determine the relevant discount factor. Even though the triangles at issue seem different in Fig. 5, it can be easily checked that the argument presented above in the context of Fig. 3 carries over *verbatim* to Fig. 5 when we substitute m_1 and m_{11} for n_4 and n_{41}. None of the formulae presented above require any modification.

The point is that the argument is of course not yet complete. Except for the plan C in Fig. 3 (and in Figs. 5 and 6), who is to say that the two straight-down-the-turnpike paths, one a truncation after two periods, are not both non-optimal? We now proceed to rule this non-optimality out. In Fig. 5, consider a program that begins at the point m_{01} and stays at the golden-rule stock thereafter, and a program that begins with the plan m_0, continues on through the MV line, and becomes the 2-period cycle at the point C. We can now determine the value of the discount factor ρ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\delta^{\rho}(x_m, \hat{x}) = \delta^{\rho}(x_m, x_m') + \frac{\rho^2}{1-\rho^2}\delta^{\rho}(\tilde{x}, (1-d)\tilde{x}) \Longrightarrow \rho^2 = \frac{\delta^{\rho}(x_m, \hat{x}) - \delta^{\rho}(x_m, x_m')}{\delta^{\rho}(\tilde{x}, (1-d)\tilde{x}) + \delta^{\rho}(x_m, \hat{x}) - \delta^{\rho}(x_m, x_m')},$$

where (x_m, x_m') is the coordinate of the plan m_0. Since $\triangle m_0 m_{01} m_{02} \simeq \triangle C c_{11} c_{12}$, we obtain

$$\rho^2 = \frac{m_{01}m_{02}}{Gc_{12} + m_{01}m_{02}} = \frac{m_{01}m_{02}}{Gc_{11}} = \frac{m_{01}m_{02}}{Cc_{11}}\frac{Cc_{11}}{Gc_{11}} = \frac{1}{\xi^2}.$$

But this allows an almost effortless completion of the argument.

To recapitulate the argument for the reader, we have shown that at the discount factor $(1/\xi)$, the plan m_0 is optimal, and that its aggregate value loss is identical to the program that begins at m_{01} and stays at the golden-rule stock thereafter, and hence the latter program is optimal. But again, this aggregate value-loss is identical to the program that begins at M, and passes through the plan M_5 to stay at the golden-rule stock thereafter, and hence this program too is optimal. Hence we have shown that any plan in the entire triangle MGG_1 (in Fig. 3) is an optimal plan. The only point that remains is the delineation optimal policy functions when the discount factor is not $(1/\xi)$. For $\rho > (1/\xi)$, we proceed just as in [5] and establish the pan-map as the optimal policy function. The case $\rho < (1/\xi)$ requires some additional work.

Towards this end, in Fig. 3, consider any plan in the segment CD', say m_1. It is clear that for any discount factor less than $(1/\xi)$, any plan on the vertical mm_{11} other than m_1 yields greater value-loss than m_1. The issue concerns subsequent plans, which is to say, the aggregate value loss of the entire program. But we can now appeal to the distinguishing characteristic of the case under consideration (the relative magnitudes of the slopes of the lines MD' and GM_4); and the general result that the value function is non-decreasing in the initial capital stock (see [4, Sect. 4, Paragraph 1]), and that therefore its dual, the aggregate value-loss, is non-increasing in the initial capital stock. This allows us to establish that the aggregate value loss of any program starting from a plan on mm_{11} other than m_1 is less than the value loss of a program that keeps to the arm MD' and the corresponding interval of the arm GM. Such a program will of course converge to the plan C, but this limiting fact is not utilized in the argument.

3.4. The Optimal Policy Correspondence

In the case $\xi(1-d) \leq 1$, as portrayed in Fig. 2, the optimal policy correspondence is given by the pan-map VGG_1D for all $\rho > 1/\xi$, by the check-map VMD for all $\rho < 1/\xi$, and by the pan-check correspondence for all $\rho = 1/\xi$.

4. The Case $(\xi - (1/\xi))(1 - d) = 1$

This case was used as the basis for the result in [4] that optimal programs in the 2-sector RSS model can be chaotic for "small" discount factors. The interest in this case, exhibited in Fig. 7, lies in the fact that, starting from a unit capital stock, optimal programs for "small" discount factors converge to the golden-rule stock in three periods.

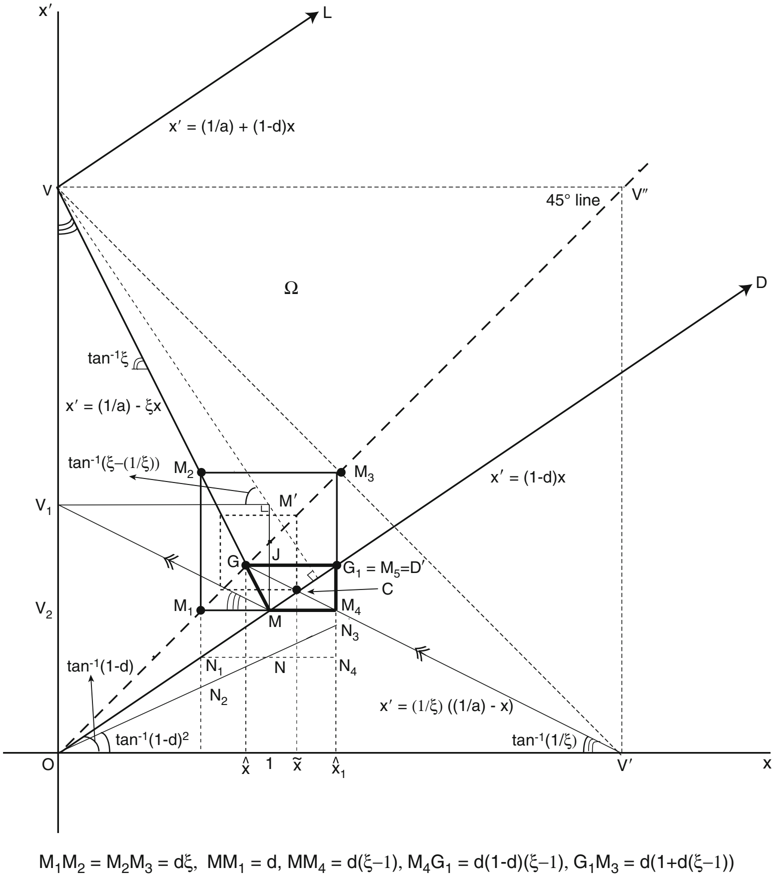

Fig. 7 The geometry of the case a$\xi^3 = (\xi^2 - 1)$ or $(\xi - (1/\xi))(1 - \mathrm{d}) = 1$

4.1. The Benchmarks

The benchmark that characterizes the case under consideration is the fact any program starting from the plan M converges in three periods, via the plans M_2 and G_1 to G; see Fig. 7. As discussed in [4], this algebraically translates to

$$(1-d)(\frac{1}{a} - \xi(1-d)) = \hat{x}.$$

We furnish a more transparent characterization of this condition.

Towards this end, let $MM_1 = x$. Then $M_1M_2 = M_2M_3 = \xi x$ and $MM_4 = (\xi - 1)x$. Now let $G_1M_4 = y$. Then $GJ = y/\xi$ and $MM_1 = (y + y/\xi) = y(1+\xi)/\xi = x$ which implies $y = x\xi/(1+\xi)$. But this yields

$$(1-d) = \frac{G_1M_4}{MM_4} = \frac{y}{(\xi-1)x} = \frac{x\xi}{(\xi-1)(1+\xi)x} = \frac{\xi}{\xi^2-1}.$$

By taking the definition of $\xi = (1/a) - (1-d)$ into account, we can rewrite this as

$$\left(\xi - \frac{1}{\xi}\right)(1-d) = 1 \Longleftrightarrow \left(\xi - \frac{1}{\xi}\right)\left(\frac{1}{a} - \xi\right) = 1 \Longleftrightarrow a = \frac{\xi^3}{\xi^2 - 1}. \quad (3)$$

A question of interest is whether (3) expresses itself in some sort of perpendicularity. To see this, consider in Fig. 7 the line MV_1 parallel to the line GV' and let the horizontal through V_1 intersect the vertical through M at M'. Then the line VM' is perpendicular to the line OD. Note that the tangent of the angle $\angle V_1M'V$ is given by VV_1/V_1M' which equals $VV_2 - V_1V_2$ which equals $\xi - (1/\xi)$. The perpendicularity follows from the fact that the angles $\angle V_1M'V$ and $\angle V_1VM'$ are complementary angles. Since $\angle VOD$ and $\angle DOV'$ are also complementary angles, and (3) yields the equality of $\angle VOD$ and $\angle V_1M'V$, the angles $\angle VOD$ and $\angle M'VO$ are also complementary angles. The argument for the perpendicularity of VM' and OD is complete.

4.2. Check-Map Dynamics

We can now use the benchmarks identified above to highlight some of the properties of the dynamics that stem from the check-map in this particular case.

As in the case considered previously, the intersection of the OD and the dual MV lines yield a 2-period cycle. However, it is easy to see that it is unstable. From the characterization in (3), we obtain

$$(1-d) - \frac{1}{\xi} = \frac{\xi}{\xi^2 - 1} - \frac{1}{\xi} = \frac{1}{\xi(\xi^2 - 1)} > 0.$$

In terms of a geometric argument based on Fig. 7, this follows from the fact that

$$(1-d) = \frac{G_1M_4}{MM_4} > \frac{G_1M_4}{GG_1} = \tan(\angle GM_4M) = \tan(\angle M_4V'O) = 1/\xi.$$

Next, we turn to 3-period cycles. The check-map has two arms, and it easy to see that the only possibility of a 3-period cycle of the order RRL, rather than LLR or RLR, where R refers to a plan on the right arm and L on the left. This implies, for an initial stock x,

$$x \to x(1-d) \to x(1-d)^2 \to \frac{1}{a} - \xi(1-d)^2x \to x \Longrightarrow \frac{1}{a} - \xi(1-d)^2x$$
$$= x \Longrightarrow (1-d)^2x = \frac{1}{a\xi} - \frac{x}{\xi}.$$

But this is nothing other than the requirement that the dual MV line intersect the line N_3 in the trapping square; see Fig. 7. And since we have already seen in Fig. 6 how to obtain a line of slope x^2 from a line of slope x, we can now turn to showing the impossibility of 3-period cycles in the case under consideration.

Towards this end, let the downward vertical from M_1 intersect the line OD at N_1 and the horizontal intersect the downward vertical from M at N and from M_1 at N_4. Let the line ON intersect the downward vertical from M_4 at N_3. It is easy to check that the slope of the line ON_3 is $tan^{-1}(1-d)^2$. We shall now show that N_3 is always below the vertex M_4. Note from Fig. 7 that

$$NN_4 = MM_4 = \frac{M_4G_1}{(1-d)} = \frac{\hat{x}-(1-d)}{(1-d)} \Longleftrightarrow N_3N_4 = \hat{x}(1-d)-(1-d)^2.$$

Since

$$M_4N_4 = M_1N_1 = (1-d)MM_1 = d(1-d),$$

we obtain

$$\begin{aligned} M_4N_4 - N_3N_4 &= M_1N_1 - N_3N_4 \\ &= d(1-d) + (1-d)^2 - \hat{x}(1-d) \\ &= (1-d)(d+1-d-\hat{x}) = (1-d)(1-\hat{x}) > 0. \end{aligned}$$

The argument is complete.

4.3. The McKenzie Bifurcation

As in Sect. 3, we have yet to establish values of the discount factor ρ for which the check map constitutes the optimal policy function. We now turn to this.

We begin with the case where the initial capital stock is $\tilde{x}$. The argument revolving around Eq. (2) applies with the relevant part of Fig. 7 magnified along the lines of Figs. 5 and 6 concerning the previous case. We conclude that with $\rho = \hat{\rho} = (1/\xi)$, the planner is indifferent between a two-period cyclical path and one in which the cycles are terminated *at any time* by his moving to the golden-rule stock and staying there.

Next, we turn to the case where the initial capital stock is unity. Again, consider two alternative paths: the first where the planner moves to the golden-rule stock and stays there (the straight-down-the-turnpike path); and the second, the path that returns to the initial capital stock after three periods. In Fig. 7, simply observe that by virtue of the similarity of the triangles, $\triangle GJM$ and $\triangle GG_1M_1$,

$$\delta^\rho(1,\hat{x}) = \rho^2\delta^\rho(\hat{x}_1,\hat{x}) \Longrightarrow \rho^2 = \frac{\delta^\rho(1,\hat{x})}{\delta^\rho(\hat{x}_1,\hat{x})} = \frac{GJ}{G_1G} = \frac{GJ}{JM}\frac{G_1M_4}{G_1G} = 1/\xi^2. \tag{4}$$

But now the argument follows along familiar lines.

At this point, the alert reader will question the argument for the cases when the discount factor is not equal to $(1/\xi)$. The fact that earlier arguments carry over *verbatim* to the case when $\rho > (1/\xi)$ is clear; the issue concerns the situation when $\rho < (1/\xi)$. The argument that needs to be adapted is the one furnished in the concluding paragraph of Sect. 3.3, one that relied crucially on the fact that the check-map ensures convergence to a 2-period cycle. The distinguishing characteristic of the current case is that no such convergence obtains, and indeed, as established in [4], the trajectories are topologically chaotic. And so one does not have recourse to an argument, especially a geometric one, that shows that the check-map to be the optimal policy function for all $\rho < (1/\xi)$. Indeed, the claim itself may be false, and we are obliged to leave it as an open question.

4.4. The Optimal Policy Correspondence

In the case $(\xi - (1/\xi))(1-d) = 1$, as portrayed in Fig. 7, the optimal policy correspondence is given by the pan-map VGG_1D for all $\rho > 1/\xi$ and by the pan-check correspondence for all $\rho = 1/\xi$.

5. The Case $(\xi - 1)(1 - d) = 1$

The interest in this case, a case original to this paper and whose geometry is exhibited in Fig. 8, lies in the fact that, starting from a unit capital stock, optimal programs for "small" discount factors exhibit three-period cycles.

5.1. The Benchmarks

The benchmark that characterizes the case under consideration is the fact any program starting from the plan M returns to it in three periods, via the plans M_2 and M_6; see Fig. 8.

A distinguishing geometric characteristic of this case is that the perpendicular to OD at M, and intersecting the Y-axis at W, leads to the segment VW being of unit length. Since $\angle MOW$ is complementary to both $\angle OWM$ and $\angle DOV'$, the latter are equal. This implies that WM_d is $1/(1-d)$ which

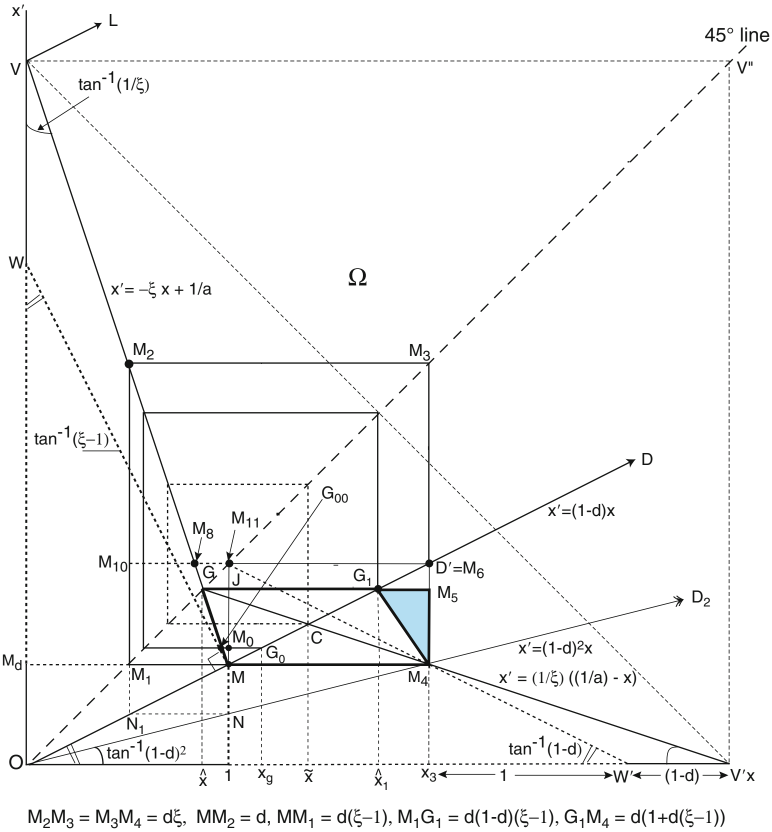

Fig. 8 The geometry of the case $(\xi - 1)(1 - \mathrm{d}) = 1$ or $\mathrm{a}(\xi^2 - \xi + 1) = (\xi - 1)$ or $(1/\mathrm{a}) - 1 = (1 - \mathrm{d}) + (1 - \mathrm{d})^{-1}$

implies, given the distinguishing characteristic of the case under consideration, that it equals[23] $(\xi - 1)$. Since OV is $1/a$, we obtain

$$VW = OV - OM_d - WM_d = \frac{1}{a} - \frac{1}{1-d} + (1-d) = \frac{1}{a} - (\xi - 1) - (1-d) = 1.$$

Next, we let OM intersect the square $M_1M_2M_3M_4$ again at M_6, and let the vertical from M intersect the 45°-line at M_{11}. We join M_{11} to M_4, and designate its intersection with the X-axis by W'. Join M_{11} to M_6. We have to show that $M_{11}M_6$ is a horizontal, or equivalently, that $MM_4M_6M_{11}$ is

[23] Recall that ξ is a positive number greater than 1.

a rectangle. But this is a straightforward deduction. Since $\triangle M_3M_2M_{11} \simeq \triangle M_3M_4M_{11}$, $M_2M_{11} = M_{11}M_4$, which is to say that $\triangle M_2M_{10}M_{11}G \simeq \triangle MM_{11}M_4$, and hence $M_{10}M_{11} = M_{11}M$, and hence $MM_{11}M_{10}M_1$ is a square. Hence $M_4M_6M_{11}M$ is a rectangle.

We may note some additional benchmarks in passing. Since MM_6 and M_4M_{11} are diagonals of the rectangle $MM_4M_6M_{11}$, $\angle M_6MM_4 = \angle M_{11}M_4M$. Furthermore, since $\angle OWM$ is complementary to $\angle WOM$, which is itself complementary to $\angle MOV'$, it is equal to $\angle MOV'$ which is equal to $\angle M_6MM_4$, and hence equal to $\angle M_{11}M_4M$, and therefore to $\angle M_4W'O$. All this establishes that both $\angle M_dWM$ and $\angle M_4W'O$ equal $\tan^{-1}(1-d)$. This means that $\triangle WMM_d \simeq \triangle 1M_{11}W'$ which in turn implies that $W'V' = (1-d)$. It also means that x_3W' is of unit length, and hence $M_4x_3 = (1-d)$.

We can use these deductions to obtain

$$\tan(\angle M_4OV') = \frac{M_4x_3}{Ox_3} = \frac{1-d}{\xi - 1} = (1-d)^2.$$

Thus the points O, L and M_4 are collinear, where N_1 is the intersection of OD and the vertical from M_1 and N is the intersection of the vertical from M and the horizontal from N_1.

In summary, the geometry of this case furnishes an important perspective on the geometric apparatus presented in the sections above. In the case depicted in Fig. 1, it is the symmetric trapezium reflecting MM_4G_1G that reflects the perpendicularity of the VM and OD lines; while in Fig. 7, it is the triangle V_1MM' and the perpendicularity of the VM' and OD lines. In the case at hand in this section, in Fig. 8, it is the rectangle $MM_4M_6M_{11}$, the perpendicularity of the WM and OD lines, and the congruence of the triangles WMM_d and $1M_{11}W'$.

5.2. Check-Map Dynamics

We can now use the benchmarks identified above to highlight some of the properties of the dynamics that stem from the check-map.

As in the case considered previously, the intersection of the OD and the dual MV lines yield a 2-period cycle. However, it is easy to see that it is unstable. Since $MM_4M_6M_{11}$ is a rectangle with each of its diagonals with slope $(1-d)$, we obtain

$$(1/\xi) = \tan(\angle GM_4M) < \tan(\angle M_{11}M_4M) = (1-d) \Longrightarrow \xi(1-d) > 1,$$

a demonstration of the instability claim as a consequence of the argumentation already appealed to above.

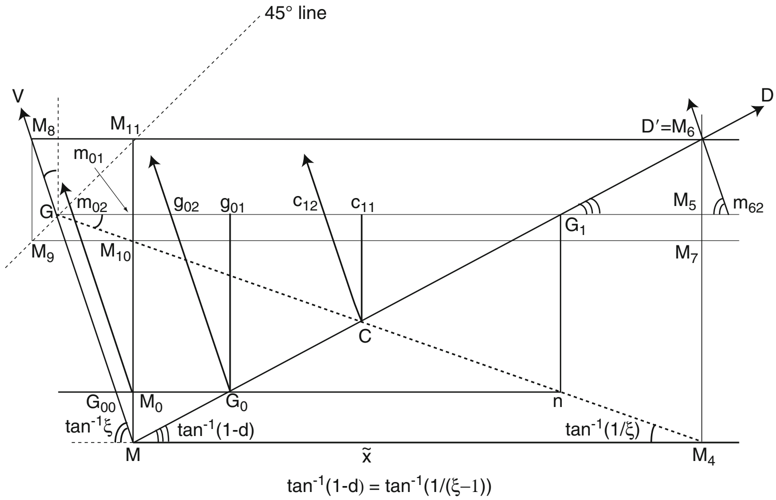

Fig. 9 Determination of ρ_1 and ρ_c in the case $(\xi - 1)(1 - \mathrm{d}) = 1$

Next, we turn to 3-period cycles. There are two alternative ways to demonstrate that there is a three-period cycle from M. For the first, simply appeal to the fact that $MM_{11}M_6M_4$ is a rectangle to directly obtain the result. For a second demonstration, appeal to the fact that the OD_2 line with slope $(1-d)^2$ intersects the trapping square $M_1M_2M_3M_4$ at M_4, and appeal to the argument made in the third paragraph of Sect. 4.2 above.

What is of particular interest is the fact this 3-period cycle is also unstable. The demonstration of this claim follows the line of argument that hinges on the dual MV line developed in Sect. 3. Towards this end, consider in Fig. 9 an initial capital stock in the vicinity of M_6, say x units to the left of the abscissa of M_6, where x is "small." The dynamics of the check-map demand that this will lead to a choice of a plan on the MV line in the interval MG_{00}, and thence through the dual MV line on a plan on the OD line with abscissa $\xi^2(1-d)x$. Since the defining characteristic of the case under consideration leads to this magnitude being $\xi^2/(\xi - 1)$, and hence to $(\xi + (\xi/(\xi - 1))$, it is clear that we end up with a plan further away from M_6 from where we began.

This argument is premised on a beginning at the left of M_6. However, any plan that begins in a " small" vicinity to the right of M_6, will fall to the left of M_6 after two periods. In passing, we leave it to the interested reader to show that the plan G_1 is also locally unstable.

5.3. Two Bifurcations

As in Sects. 3 and 4, we have yet to establish values of the discount factor ρ for which the check map constitutes the optimal policy function. We now turn to this.

Consider, as above, the case where the initial capital stock is $\tilde{x}$, and appeal to the argument offered therein that with $\rho = \hat{\rho} = (1/\xi)$, the planner is indifferent between a two-period cyclical path and one in which the cycles are terminated *at any time* by his moving to the golden-rule stock and staying there; see Figs. 8 and 9.

Next, turn to the case where the initial capital stock is unity. Again, consider two alternative paths: the first where the planner moves to the golden-rule stock and stays there (the straight-down-the-turnpike path); and the second, the path that returns to the initial capital stock after three periods. In terms of Figs. 8, the path that moves from J to G (m_{01} to G in Fig. 9) compared to the path MM_2M_6M. We now determine the value of the discount factor ρ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\begin{aligned} \delta^{\rho}(1, \hat{x}) &= \delta^{\rho}(\bar{x}, (1-d)\bar{x})(\rho^2 + \rho^5 + \cdots) \\ \Longrightarrow \frac{\delta^{\rho}(1, \hat{x})}{\delta^{\rho}(\bar{x}, (1-d)\bar{x})} &= \frac{\rho^2}{1-\rho^3} \end{aligned} \tag{5}$$

Now, again by Eq. (5), and with reference to Fig. 9, we obtain

$$\frac{\delta^{\rho}(1, \hat{x})}{\delta^{\rho}(\bar{x}, (1-d)\bar{x})} = \frac{Gm_{01}}{Gm_{62}} = \frac{Gm_{01}}{M_8M_6} = \frac{Gm_{01}}{M_8M_{11} + M_{11}M_6}.$$

Next, by focussing on triangles $\triangle MM_{11}M_8$ and $\triangle MM_{11}M_6$, we obtain

$$\begin{aligned} &\frac{M_8M_{11}}{MM_{11}} = \frac{1}{\xi}, \quad \frac{MM_{11}}{M_{11}M_6} = 1-d \text{ and} \\ &\frac{Gm_{01}}{MM_{11} - Gm_{01}} = \frac{1}{\xi} \Longrightarrow \frac{Gm_{01}}{MM_{11}} = \frac{1}{1+\xi}. \end{aligned}$$

On making the appropriate substitutions, and on eliminating MM_{11}, we obtain

$$\begin{aligned} \frac{Gm_{01}}{M_8M_{11} + M_{11}M_6} &= \frac{\frac{MM_{11}}{1+\xi}}{\frac{MM_{11}}{\xi} + \frac{MM_{11}}{1-d}} = \frac{\xi(1-d)}{(1+\xi)(\xi + (1-d))} \\ &= \frac{\xi}{(1+\xi)(1-\xi+\xi^2)} \equiv \tau. \end{aligned}$$

We thus obtain

$$\frac{\rho^2}{1-\rho^3} = \tau \Longrightarrow f(\rho) = \rho^3 + (1/\tau)\rho^2 - 1 = 0.$$

Thus the root ρ_c of Eq. (5) is now seen to be one that leads to the solution of the equation $f(\rho) = 0$.

It is easily checked that $f(\cdot)$ is a monotonically increasing continuous function that takes the value -1 at $\rho = 0$, and the value $(1/\tau)$ at $\rho = 1$. Thus it has a unique root ρ_c. The interesting question is how this root relates to $\hat{\rho}$. But this can be easily checked as follows:

$$f(\hat{\rho}) = f(\frac{1}{\xi}) = \frac{1}{\xi^3} + \frac{(1+\xi)(1-\xi+\xi^2)}{\xi^3} - 1 = \frac{2}{\xi^3} > 0.$$

Hence $\hat{\rho} > \rho_c$.

Now consider another path alternative to the straight-down-the-turnpike path; namely, the path that does not return to the initial capital stock after three periods but one that moves to the golden-rule stock in the third period. In terms of Fig. 9, the path that moves from $m_{01}$to G, compared to the path $MM_2M_6m_{01}$. We now determine the value of the discount factor $\bar{\rho}$ that equates the aggregate value losses of these two paths. This is to say that we want the root to the equation

$$\delta^{\rho}(1,\hat{x}) = \rho^2\delta^{\rho}(\bar{x},(1-d)\bar{x}) + \rho^3\delta^{\rho}(1,\hat{x})$$

$$\Longrightarrow \frac{\delta^{\rho}(1,\hat{x})}{\delta^{\rho}(\bar{x},(1-d)\bar{x})} = \frac{\rho^2}{1-\rho^3} = \tau \tag{6}$$

which is identical to Eq. (5).

But now we can consider another alternative to the straight-down-the-turnpike path; namely, the path that moves to the golden-rule stock in the $(3n)$th-period after cycling n times, where $n \in \mathbb{N}$. Let the value of the discount factor that equates the aggregate value losses of these two paths be indicated simply by ρ. This is the root to the equation

$$\delta^{\rho}(1,\hat{x}) = \rho^2\delta^{\rho}(\bar{x},(1-d)\bar{x})(1+\rho^3+\cdots+\rho^{(3n-1)}) + \rho^{3n}\delta^{\rho}(1,\hat{x})$$
$$= \frac{\rho^2(1-\rho^{3n})}{1-\rho^3}\delta^{\rho}(\bar{x},(1-d)\bar{x}) + \rho^{3n}\delta^{\rho}(1,\hat{x}).$$

This implies

$$\frac{\delta^{\rho}(1,\hat{x})}{\delta^{\rho}(\bar{x},(1-d)\bar{x})}=\frac{\rho^2}{1-\rho^3},$$

which is again identical to Eq. (5). Hence $\rho=\rho_c$ for all $n\in\mathbb{N}$.

In Fig. 9, let M_{10} be the intersection of M_4G with MM_{11}. Since M_{10} is on M_4G, a path starting from M_{10} returns to M_{10} in two periods. Now compare the straight-down-the-turnpike path with the full-employment path starting at M_{10} that moves to the golden-rule stock in the second period. The value of the discount factor that equates the aggregate value losses of these two paths is given by the root to the equation

$$Gm_{01}=Gm_{12}+\rho^2 Gm_{01}\Longrightarrow\frac{Gm_{01}}{Gm_{12}}=\frac{1}{1-\rho^2},\tag{7}$$

where m_{12} is the point of intersection, not shown in Fig. 9, of GG_1 with a line parallel to MM_8 through M_{10}. Now observe that

$$\frac{Gm_{01}}{Gm_{12}}=\frac{Gm_{01}}{Gm_{01}-m_{12}m_{01}}=\left(1-\frac{m_{12}m_{01}}{Gm_{01}}\right)^{-1}=\left(1-\frac{(M_{10}m_{01})/\xi}{(M_{10}m_{01})\xi}\right)^{-1}$$
$$=\left(1-\frac{1}{\xi^2}\right)^{-1}.$$

We have shown that the root to the equation is $(1/\xi)$.

Next, compare the straight-down-the-turnpike path with the full-employment path that keeps oscillating between M_{10} and M_8. The discount factor is given by the root to the equation

$$Gm_{01}=Gm_{12}(1+\rho^2+\cdots)\Longrightarrow\frac{Gm_{01}}{Gm_{12}}=\frac{1}{1-\rho^2},\tag{8}$$

which is identical to the equation already considered above, and thus the discount factor is unchanged at $(1/\xi)$.

Finally, in this connection, compare the straight-down-the-turnpike path with the full-employment path that keeps oscillating between M_i and M_7 and moves to the golden-rule stock in the $(2n+1)$th-period, where $n\in\mathbb{N}$. It is of interest that the discount factor that equates the aggregate value losses of these two paths is also $(1/\xi)$. To see this, we need to consider the root to the equation

$$Gm_{01}=Gm_{12}(\rho^2+\cdots+\rho^{2n})+\rho^{(2n+2)}Gm_{01}\Longrightarrow\frac{Gm_{01}}{Gm_{12}}=\frac{1}{1-\rho^2}.\tag{9}$$

Next, in Fig. 9, let G_0 be the point of intersection of the horizontal from the point n at which the vertical from G_1 intersects GM_4. Certainly, a path starting from G_0 returns to G_1 in two periods. We now compare this path with the straight-down-the-turnpike path. The relevant discount factor is given by the root of the following equation.

$$Gg_{01} = Gg_{02} + \rho^2 G_1 G \Longrightarrow g_{02}g_{01} = \rho^2 G_1 G$$
$$\Longrightarrow \rho^2 = \frac{g_{02}g_{01}}{G_1 G} = \frac{(G_0 g_{01})/\xi}{\xi(G_0 g_{01})} = \frac{1}{\xi^2}. \tag{10}$$

We have shown that the root to the equation is $(1/\xi)$.

The interesting argument relates to the comparison of that begins at the plan M_0 in Figs. 8 and 9. Note that this allows the feasibility of a program that makes a value-loss in the first period, and after a further value-loss at G_1, converges to the golden-rule stock. The question at issue is the discount factor at which the aggregate value-losses of such a path, hereafter path S, is equal to the path that starts at the unit capital stock and returns to it after three periods. We shall furnish a geometric proof for the underlying polynomial.

$$Gm_{02} + \rho^2 GG_1 = m_{62}G(\rho^2 + \rho^5 + \cdots) = m_{62}G\frac{\rho^2}{1-\rho^3}. \tag{11}$$

But this leads to

$$\begin{aligned}0 &= Gm_{02} - \rho^3 Gm_{02} + \rho^2 GG_1 - \rho^5 GG_1 - \rho^2 Gm_{62}\\ &= -(\rho^5 GG_1 + \rho^3 Gm_{02} + \rho^2 Gm_{62} - Gm_{02})\\ &= -\rho^5\xi^2(\xi + (1-d) + \rho^3 Gm_{02}(1/\xi) + (\xi + (1-d))\rho^2 - (1/\xi)).\end{aligned}$$

We thus obtain the fundamental polynomial for the second bifurcation to be

$$\xi^2\rho^5 + a\rho^3 + \xi\rho^2 - a = 0.$$

We check by inspection that one of the roots of this polynomial is $(1/\xi)$, and therefore, by division, and the appropriate substitution of a by $(\xi - 1)/(\xi^2 - \xi + 1)$, we obtain the quartic

$$\xi(\xi^2 - \xi + 1)\rho^4 - (\xi^2 - \xi + 1)\rho^3 + \xi\rho^2 + \xi(\xi - 1)\rho - (\xi - 1) = 0.$$

On considering the specific case of $\xi = 3$ and therefore $d = 1/2$ and $a = 2/7$, we obtain the polynomial

$$21\rho^4 - 7\rho^3 + 3\rho^2 + 6\rho - 2 = 0,$$

with a value of $\hat{\rho}_1$ to be 299/1018 = 0.29371322080902. If we at this discount factor, we compute the value of consumption from the 3-period cycle, and the path that begins at the plan r and ends up at the golden-rule stock after two periods, we find it to be 1.26518108874487 in either case.

Note that if we compare the value-losses from the path S to a path that begins as the 3-period cycle from the unit capital stock, but instead of completing the cycle, is made to converge to the golden-rule stock, the corresponding equation for the discount factor is given by

$$Gr_{12} + \rho^2 GG_1 = \rho^2 m_{62} G + \rho^3 Gr_{12} + \rho^5 GG_1,$$

which is identical to Eq. (11).

So far so good. However, just as in the case considered in Sect. 4, the difficult issues arise for the situation when $0 < \rho < \rho_c$. One would expect that the optimal policy correspondence would be given by the check-map, but in the absence of convergence, the geometric methods being presented in this paper do not suffice, and we are obliged to leave the complete characterization as an open problem.

5.4. The Optimal Policy Correspondence

In the case $(\xi - 1)(1 - d) = 1$, as portrayed in Figs. 8 and 9, the optimal policy correspondence is given by the pan-map VGG_1D for all $\rho > 1/\xi$, by the pan-map $VG_{00}G_0D$ for all $\rho_c < \rho < 1/\xi$, and by the pan-pan correspondence for all $\rho = 1/\xi$, and the pan-check correspondence for all $\rho = \rho_c$.

6. Concluding Observation

In this paper, we have provided a substantial geometric apparatus that goes well beyond that presented in [5, 8]. Our progress from a substantive point can be gauged by a quotation from the concluding remarks in [9]:

> The bifurcation result naturally raises two related questions. First, what kind of optimal behavior would one observe at the bifurcation value of the discount factor $\rho^* = (1/\xi)$? Second, what is the optimal policy correspondence when $\rho < (1/\xi)$? It is possible that ... the analogy is complete with the discounted case. ... However, a more intricate picture is also possible.

In addition to providing intuitive geometric arguments, we have made substantial progress regarding this picture when ξ lies in the interval $-1 < \xi < 1/(1 - d)$ and for two of its specific values. Whereas there is little further to be said for the former, it is clear that a more detailed comparative analysis of the two latter cases remains to be done. This would involve the delineation of the optimal policy correspondence for the *entire* range of the discount factor, as well as a better understanding of the role these cases play in the investigation of exact parametric restrictions for the existence of chaotic programs. We defer such an investigation to future work.

References

1. Cass, D., Stiglitz, J.E.: The implications of alternative savings and expectation hypotheses for choices of technique and patterns of growth. J. Polit. Econ. **77**, 586–627 (1970)
2. Fujio, M.: Optimal Transition Dynamics in the Leontief Two-sector Growth Model, Unpublished Ph.D. dissertation, The Johns Hopkins University (2006)
3. Khan M.A., Mitra, T.: On choice of technique in the Robinson-Solow-Srinivasan model. Int. J. Econ. Theor. **1**, 83–110 (2005)
4. Khan M.A., Mitra, T.: On topological chaos in the Robinson-Solow-Srinivasan model. Econ. Lett. **88**, 127–133 (2005)
5. Khan M.A., Mitra, T.: Discounted optimal growth in the two-sector RSS model: a geometric investigation. Adv. Math. Econ. **8**, 349–381 (2006)
6. Khan M.A., Mitra, T.: Undiscounted optimal growth under irreversible investment: a synthesis of the value-Loss approach and dynamic programing. Econ. Theor. **29**, 341–362 (2006)
7. Khan M.A., Mitra, T.: Optimal Cyclicity and Chaos in the 2-Sector RSS Model: A Constructive Synthesis. Cornell University, Mimeo (2006)
8. Khan M.A., Mitra, T.: Optimal growth in a two-sector RSS model without discounting: a geometric investigation. Jpn. Econ. Rev. **58**, 191–225 (2007)
9. Khan M.A., Mitra, T.: Optimal growth under discounting in the two-sector Robinson-Solow-Srinivasan model: a dynamic programing approach. J. Differ. Equat. Appl. **13**, 151–168 (2007)
10. Khan M.A., Mitra, T.: Complicated Dynamics and Parametric Restrictions in the Robinson-Solow-Srinivasan Model. Cornell University, Mimeo (2010)
11. McKenzie, L.W.: Optimal Economic Growth, Turnpike Theorems and Comparative Dynamics. In: Arrow, K.J., Intrilligator, M. (eds.) Handbook of Mathematical Economics, vol. 3, pp. 1281–1355. North-Holland, New York (1986)
12. Mitra, T.: Characterization of the Turnpike property of optimal paths in the aggregative model of intertemporal allocation. Int. J. Econ. Theor **1**, 247–275 (2005)
13. Radner, R.: Paths of economic growth that are optimal only with respect to final states. Rev. Econ. Stud. **28**, 98–104 (1961)
14. Stiglitz, J.E.: A note on technical choice under full employment in a socialist economy. Econ. J. **78**, 603–609 (1968)
15. Stiglitz, J.E.: The badly behaved economy with the well-behaved production function. In: Mirrlees, J.A., Stern, N.H. (eds.) Models of Economic Growth, Chap. 6. Wiley, New York (1973)